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NOTE

ON LABELED VERTEX-TRANSITIVE DIGRAPHS WITH A PRIME NUMBER OF VERTICES

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We enumerate, up to isomorphism, several classes of labeled vertex-transitive digraphs with a prime number of vertices.

There are many unsolved enumeration problems stated in [5]. Recently, Robinson in [8] posed more enumeration problems. Here, we give some partial answers to the problems posed on p. 181 in [8], i.e., by using some of the results in [3] and [4], we enumerate several classes of labeled vertex-transitive digraphs with a prime number, p, of vertices; specifically, we count those that are symmetric (both vertex-transitive and edge-transitive), those with a given group of automorphisms and those that are self-complementary. Roughly, our method is to use the group of automorphisms, Pólya's and de Bruijn's enumeration theorems and the following lemma.

Lemma 1 (See [5, Chapter 1]). *Let $H_1, H_2, \ldots, H_t$ be all nonisomorphic digraphs (graphs) with n vertices and $|\Gamma(H_i)|$ be the order of the automorphism group $\Gamma(H_i)$ of H_i for $i = 1, 2, \ldots, t$. Then the total number of distinct labeled digraphs (graphs) with n vertices is $\sum_{i=1}^{t} n!/|\Gamma(H_i)|$.*

Theorem 1. *The number of labeled symmetric digraphs on a prime number p vertices is*

$$2+\sum_{n}\frac{(p-1)!}{n}$$

where n is a divisor of $p-1$, and $1 \leq n < p-1$.

Proof. In [3], Theorem 3 states that the group of automorphisms $\Gamma(X)$ of a non-null and non-complete vertex-transitive digraph X on p vertices is generated by R and σ with defining relations $R^p = e$, $\sigma^n = e$ and $\sigma^{-1}R\sigma = R^r$ where e is the identity of $\Gamma(X)$, n divides $p-1$, $1 \leq n < p-1$ and $r^n \equiv 1 \pmod p$. If the digraph

X is symmetric, then, by Corollary 3.1 in [3], $|\Gamma(X)| = np$ where n is the degree of X.

By Lemma 1, including the null and complete graphs with p vertices, the total number of labeled symmetric digraphs with p vertices is

$$2+\sum_n \frac{p!}{np} = 2+\sum_n \frac{(p-1)}{n}$$

where n is a divisor of $p-1$, and $1 \leq n < p-1$.

In Theorem 4 of [4] we used the group structure described in the preceding proof to enumerate the non-isomorphic digraphs with a prime number p of vertices and with group of automorphisms $\geq \langle R, \sigma \rangle$, where $\sigma = \tau^m$ and $\tau^{p-1} = e$. This enumerating function is

$$\frac{1}{m}\sum_d \phi(d)(1+x^{dnp})^{m/d} \tag{1}$$

where the sum is taken over the divisors d of m, $mn = p-1$ and where ϕ is Euler's ϕ function.

Lemma 2. *The enumerating function for the non-isomorphic classes of vertex-transitive digraphs with a prime number of vertices and with the group of automorphisms $\langle R, \tau^m \rangle$ is*

$$\sum_D \mu\left(\frac{m}{D}\right) \cdot \frac{1}{D} \sum_d \phi(d)(1+x^{dnp})^{D/d} \tag{2}$$

where m divides $p-1$, $Dn = (p-1)$, the first summation is taken over the divisors D of m and the second over the divisors d of D, μ is the Möbius function.

We omit the proof here. Our Lemma 2 is a generalization of Theorem 5 in [4]. There we enumerated the digraphs with p vertices and with the group of automorphisms $\langle R, \tau^{p-1} \rangle = \langle R \rangle$. Here we apply the same exclusion–inclusion principle to the group of automorphisms $\langle R, \tau^m \rangle$.

Example 1. For $p = 13$ and $m = 6$, we enumerate the digraphs with 13 vertices each of whose group of automorphisms is $\langle R, \tau^6 \rangle$. By (2), we have

$$\begin{aligned} &\tfrac{1}{6}\sum_{d\mid 6} \phi(d)(1+x^{26d})^{6/d} - \tfrac{1}{3}\sum_{d\mid 3} \phi(d)(1+x^{52d})^{3/d} \\ &\quad -\tfrac{1}{2}\sum_{d\mid 2} \phi(d)(1+x^{78d})^{2/d} + (1+x^{156}) = x^{26} + 2x^{52} + 3x^{78} + 2x^{104} + x^{130}. \end{aligned}$$

Thus, on 13 vertices there are 9 nonisomorphic digraphs with the group of automorphisms $\langle R, \tau^6 \rangle$: one has 26 edges, 2 have 52 edges, etc. Since the order of the group $\langle R, \tau^6 \rangle$ is $2 \cdot 13$, by Lemma 1, the number of labeled digraphs on 13 vertices having the group $\langle R, \tau^6 \rangle$ is $9 \cdot 13!/2 \cdot 13 = 54 \cdot 11!$.

Replacing x by 1 in (2) and applying Lemma 1, we have

Theorem 2. *The number of labeled digraphs with a prime p vertices and with the group of automorphisms $\langle R, \tau^m \rangle$ is*

$$\left[\sum_D \mu\left(\frac{m}{D}\right) \cdot \frac{1}{D} \sum_d \phi(d) \cdot 2^{D/d}\right] m(p-2)! \tag{3}$$

where m divides $p-1$, the first summation is taken over the divisor D of m and the second summation is taken over the divisor d of D.

A vertex-transitive digraph is said to be strongly vertex-transitive if its group of automorphisms is cyclic.

Corollary 2.1. *The number of labeled strongly vertex-transitive digraphs with a prime number p vertices is*

$$\left[\sum_{D \mid (p-1)} \mu\left(\frac{p-1}{D}\right) \frac{1}{D} \sum_{d \mid D} \phi(d) \cdot 2^{D/d}\right] (p-1)! \tag{4}$$

Proof. It follows from (3) with $m = p-1$.

Finally, we shall count the labeled self-complementary digraphs with p vertices each of whose group is $\langle R, \tau^m \rangle$. We showed in [3] that a digraph with p vertices and with the group of automorphisms $\langle R, \tau^m \rangle$ has degree a multiple of $(p-1)/m$, since τ^m fixes one vertex and is a collection of cycles of length $(p-1)/m$. A self-complementary vertex-transitive digraph with p vertices has degree $\frac{1}{2}(p-1)$. Thus, $\frac{1}{2}(p-1)$ is a multiple of $(p-1)/m$ and m must be even.

Theorem 3. *The number of labeled vertex-transitive self-complementary digraphs with a prime number p vertices and with the group of automorphisms $\langle R, \tau^m \rangle$ is*

$$\left[\sum_D \frac{\mu(m/d)}{D} \sum_d \phi(d) \cdot 2^{D/d}\right] m(p-2)! \tag{5}$$

where m is an even divisor of $p-1$, the first summation is taken over the even divisors D of m and the second summation is taken over the even divisors d of D.

Proof. We apply de Bruijn's Theorem [1] to the enumeration (2) of digraphs each of whose group is $\langle R, \tau^m \rangle$. By using a second permutation group which places each digraph in the same class with its complement as in [4] and [7], we are able to determine the number of those that are self-complementary. This number is

$$\sum_D \frac{\mu(m/D)}{D} \sum_D \phi(d) \frac{\partial^{D/d}}{\partial z_d^{D/d}} e^{2(z_2+z_4+\cdots)}, \tag{6}$$

evaluated at $z_1 = z_2 = \cdots = 0$. Since only even values of d result in non-zero partial derivatives, we combine (6) with Lemma 1 to obtain (5).

Corollary 3.1. *The number of labeled strongly vertex-transitive self-complementary digraphs with a prime number p vertices is*

$$\left[\sum_{D \mid p-1} \frac{\mu((p-1)/D)}{D} \sum_{d \mid D} \phi(d) \cdot 2^{D/d}\right](p-1)! \tag{7}$$

where D and d are even.

Proof. It follows from (5) by letting $m = p-1$.

Corollary 3.2. *The total number of labeled vertex-transitive self-complementary digraphs with a prime number p of vertices is*

$$\sum_{m \mid p-1} \left[\sum_{D \mid m} \frac{\mu(m/D)}{D} \sum_{d \mid D} \phi(d) \cdot 2^{D/d}\right] m(p-2)! \tag{8}$$

where m, D and d are even.

Proof. The total number is obtained by summing (5) over all even m.

Example 2. For $p = 13$ we count the self-complementary vertex-transitive digraphs with 13 vertices and with the group of automorphisms $\langle R, \tau^m \rangle$. For $m = 12$,

$$\tfrac{1}{12} \sum_{d \mid 12} \phi(d) \cdot 2^{12/d} - \tfrac{1}{6} \sum_{d \mid 6} \phi(d) \cdot 2^{6/d}$$

$$-\tfrac{1}{4} \sum_{d \mid 4} \phi(d) \cdot 2^{4/d} + \tfrac{1}{2} \cdot 2 = 8 - 2 - 2 + 1 = 5.$$

For $m = 6$,

$$\tfrac{1}{6} \sum_{d \mid 6} \phi(d) \cdot 2^{6/d} - \tfrac{1}{2} \ 2 = 2 - 1 = 1.$$

For $m = 4$,

$$\tfrac{1}{4} \sum_{d \mid 4} \phi(d) \cdot 2^{4/d} - \tfrac{1}{2} \cdot 2 = 2 - 1 = 1.$$

For $m = 2$,

$$\tfrac{1}{2} \cdot \phi(2) \cdot 2 = 1.$$

By (8), the total number of labeled self-complementary vertex-transitive digraphs on 13 vertices is

$$60 \cdot 11! + 6 \cdot 11! + 4 \cdot 11! + 2 \cdot 11! = 72 \cdot 11!$$

Finally, we observe that the total number of all labeled vertex-transitive digraphs with a prime number p vertices is obtained by summing (3) for all $m > 1$ such that m divides $p-1$, and adding 2 to include the labeled null and complete graphs.

Note added in proof

A formula similar to our result in Lemma 2 was presented by Brian Alspach in "Point-symmetric graphs and digraphs of prime order and transitive permutation groups of prime degree", J. Combin. Theory (B) 15 (1973) 12–17.

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